An Extension of Schwarzschild Space to $r = 0$

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Abstract

A more rigorous treatment of the Schwarzschild metric by making use of the energy-momentum tensor of a single point particle as source term shows that

$$g_{00} = -\{1 - \tfrac{2GM}{c^2 r} - \tfrac{8G^2M^2}{c^4 r^2}(\theta(r) - 1)\} \exp[2(\theta(r) - 1)]$$
$$g_{rr} = \{1 - \tfrac{2GM}{c^2 r} - \tfrac{8G^2M^2}{c^4 r^2}(\theta(r) - 1)\}^{-1}$$

The existence of a discontinuity at $r = 0$ leads to an infinite repulsive force that will change the ultimate fate of a free fall test particle to a bouncing state.

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We intend to show that at $r=0$ there is a mathematical inconsistency in derivation of the Schwarzschild metric which is the static solution of Einstein field equations for vacuum space around a point mass M[1] . It is a simple case that is propounded in any related text in GR. Usually similar approaches to the problem are followed[2-6]. We use the notations and derivations of Weinberg[2] for the sake of definiteness.
The standard form for general isotropic metric is :

$$d\tau^2 = B(r)dt^2 - A(r)dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\varphi^2 \tag{1}$$

The components of Ricci tensor for the metric (1) are:

$$R_{rr} = \frac{B''}{2B} - \frac{B'}{4B}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{A'}{rA} \tag{2}$$

$$R_{\theta\theta} = -1 + \frac{r}{2A}\left(-\frac{A'}{A} + \frac{B'}{B}\right) + \frac{1}{A} \tag{3}$$

$$R_{\varphi\varphi} = \sin^2\theta\ R_{\theta\theta} \tag{4}$$

$$R_{tt} = -\frac{B''}{2A} + \frac{B'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{B'}{rA} \tag{5}$$

$$R_{\mu\nu} = 0 \qquad for\ \mu \neq \nu$$

Prime means differentiation with respect to r. The applied field equation for empty space is

$$R_{\mu\nu} = 0 \tag{6}$$

Using (6) we may conclude that $\frac{R_{rr}}{A} + \frac{R_{tt}}{B} = 0$, and by inserting (2) and (5) in this relation we get

$$-\frac{1}{rA}(\frac{A^{'}}{A}+\frac{B^{'}}{B})=0 \tag{7}$$

Integration of (7) with respect to r and imposing Minkowski metric as r tends to infinity for boundary condition gives

$$A=\frac{1}{B} \tag{8}$$

Now by arranging (2) ,(3), (6) and (8) we obtain

$$R_{\theta\theta}=-1+rB^{'}+B=0 \tag{9}$$

$$R_{rr}=\frac{B^{''}}{2B}+\frac{B^{'}}{rB}=0 \tag{10}$$

Integrating (9) with respect to r gives

$$rB=r+const. \tag{11}$$

For fixing the constant of integration in (11) we recall that at great distances from mass M,the component of $g_{tt}=-B$ must approach to $-1-\frac{2\Phi}{c^2}$, where Φ is the Newtonian potential $-\frac{GM}{r}$. Hence it is equal to $-\frac{2GM}{c^2}$ and our final result is

$$B=1-\frac{2GM}{c^2r} \tag{12}$$

According to the correspondence principle general relativity must agree on the one hand with special relativity in the absence of gravitation and on the other hand with Newtonian theory of gravitation in the limit of weak gravitational fields and low velocities. The corresponding field equation of this problem in Newtonian gravity is the Poisson equation

$$\nabla^2\Phi=4\pi GM\delta(\overrightarrow{r})$$

It is remarkable to notice that this equation is defined in the whole space even at $r=0$ and its solution which is proportional to inverse of r satisfies the field everywhere even at $r=0$. Thus we have right to expect that the general relativity field equations which are supposed to be yielded to this equation at weak field limit must be well defined at $r = 0$ [7,8].

Since the singularity at $r = 0$ is not a hypothetical case and it is the inevitable end state of gravitational collapse of a massive body , the external gravitational field of our point particle must somehow be traced to the source located at $r = 0$. We consider this classical problem not in the belief that it is the final word , but rather a useful start which quantum phenomena will no doubt modify. It can then be used as a background for the more advanced theories. Such advanced theories are important , but they have achieved their greatest successes when the corresponding classical problem has been fully understood.

The important point is that for this case $T_{\mu\nu}$ does not vanish everywhere, it is nonzero at r exactly equal to zero. For finding the $T_{\mu\nu}$ of a mass point M located at x_1 we may use the action [9]

$$I_M = -M \int_{-\infty}^{+\infty} dp[-g_{\mu\nu}(x_1) \frac{dx_1^\mu(p)}{dp} \frac{dx_1^\nu(p)}{dp}]^{\frac{1}{2}} \tag{13}$$

$T^{\mu\nu}$ is defined to be

$$\delta I_M = \frac{1}{2} \int d^4x \sqrt{g(x)} T^{\mu\nu}(x) \delta g_{\mu\nu}(x) \tag{14}$$

A straight forward calculation gives

$$\delta I_M = \frac{1}{2} M \int_{-\infty}^{+\infty} dp[-g_{\mu\nu}(x_1(p)) \frac{dx_1^\mu(p)}{dp} \frac{dx_1^\nu(p)}{dp}]^{-\frac{1}{2}} \frac{dx_1^\lambda(p)}{dp} \frac{dx_1^\kappa(p)}{dp} \delta g_{\lambda\kappa}(x_1(p)) \tag{15}$$

This is of the form (14) with

$$T^{\lambda\kappa}(x) = g^{-\frac{1}{2}}(x) M \int_{-\infty}^{+\infty} d\tau_1 \frac{dx_1^\lambda}{d\tau_1} \frac{dx_1^\kappa}{d\tau_1} \delta^4(x - x_1) \tag{16}$$

if we take $\int d^4x \delta^4(x - x_1) = 1$ as is taken by Weinberg. It would be more concrete if we take $\int \sqrt{g} d^4x \delta^4(x - x_1) = 1$ which in this case we come to

$$T^{\lambda\kappa}(x) = M \int_{-\infty}^{+\infty} dp[-g_{\mu\nu}(x_1) \frac{dx_1^\mu}{dp} \frac{dx_1^\nu}{dp}]^{-\frac{1}{2}} \frac{dx_1^\lambda(p)}{dp} \frac{dx_1^\kappa(p)}{dp} \delta^4(x - x_1(p)) \tag{17}$$

For a partcile located at the origin we have $x_1^\mu = (x_1^0, 0)$, and in a frame at rest with respect to the particle the components of the four velocity are

$$u^i = 0 \ , \ u^t = B^{-\frac{1}{2}} \ , \ u_t = -B^{\frac{1}{2}} \tag{18}$$

Thus the (00)-component of (17) is

$$\begin{aligned} T^{00}(x) &= M \int_{-\infty}^{+\infty} dp[B(0)]^{-\frac{1}{2}} \frac{dx_1^0}{dp} \delta^4(x - x_1) \\ &= M \int_{-\infty}^{+\infty} dx_1^0 \frac{\delta^4(x - x_1)}{\sqrt{B(0)}} \end{aligned} \tag{19}$$

To find out the form of $\delta^4(x - x_1)$ it is more convenient to work in the coordinate system $x^\mu = (x^0, r, q = cos\theta, \varphi)$. In this system we have $g_{qq} = \frac{g_{\theta\theta}}{sin^2\theta}$ and $\sqrt{g} = r^2\sqrt{AB}$. Thus we may write

$$\int r^2\sqrt{AB}\delta^4(x - x_1)d^4x = 1 \tag{20}$$

(20) will be satisfied if we take

$$\delta^4(x - x_1) = \frac{\delta(x^0 - x_1^0)\delta(r)}{2\pi r^2\sqrt{AB}} \tag{21}$$

where we have $\int_0^{+\infty} \delta(r)dr = \frac{1}{2}$.

Inserting (21) in (19) gives

$$T^{00}(x) = \frac{M\delta(r)}{2\pi r^2\sqrt{ABB(0)}} \tag{22}$$

We should notice that $T^{tt} = c^2T^{00}$ and $T^{\tau\tau} = BT^{tt}$. Thus we have

$$T^{tt}(x) = \frac{Mc^2\delta(r)}{2\pi r^2\sqrt{ABB(0)}} \tag{23}$$

and

$$T^{\tau\tau} = \frac{Mc^2\delta(r)}{2\pi r^2}\sqrt{\frac{B}{AB(0)}} \tag{24}$$

Using (24) we may write

$$\int T^{\tau\tau} 4\pi r^2 \sqrt{A} dr = Mc^2 \tag{25}$$

as we should have expected. Also we have

$$T_{00}(x) = \frac{Mc^2\delta(r)}{2\pi r^2} B \sqrt{\frac{B}{AB(0)}} \tag{26}$$

Then $T \equiv g^{\mu\nu} T_{\mu\nu}$ is

$$T = -\frac{Mc^2\delta(r)}{2\pi r^2} \sqrt{\frac{B}{AB(0)}} \tag{27}$$

Now the components of $S_{\mu\nu}$ which are defined as $S_{\mu\nu} = T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T$ are:

$$S_{tt} = \frac{Mc^2\delta(r)}{4\pi r^2} B \sqrt{\frac{B}{AB(0)}} \tag{28}$$

$$S_{rr} = \frac{Mc^2\delta(r)}{4\pi r^2} \sqrt{\frac{AB}{B(0)}} \tag{29}$$

$$S_{\theta\theta} = \frac{Mc^2\delta(r)}{4\pi \sqrt{\frac{B}{AB(0)}}} \tag{30}$$

$$S_{\varphi\varphi} = S_{\theta\theta} \sin^2\theta \tag{31}$$

$$S_{\mu\nu} = 0 \qquad \mu \neq \nu \tag{32}$$

The rigorous treatment of the problem is to solve the following equations

$$R_{\mu\nu} = -\frac{8\pi G}{c^4} S_{\mu\nu} \tag{33}$$

where the components of $S_{\mu\nu}$are given by (28)-(32). Using (33) and (2)-(5) we obtain

$$R_{rr} = \frac{B^{''}}{2B} - \frac{B^{'}}{4B}(\frac{A^{'}}{A} + \frac{B^{'}}{B}) - \frac{A^{'}}{rA} = -\frac{2GM\delta(r)}{c^2r^2}\sqrt{\frac{AB}{B(0)}} \tag{34}$$

$$R_{\theta\theta} = -1 + \frac{r}{2A}(-\frac{A^{'}}{A} + \frac{B^{'}}{B}) + \frac{1}{A} = -\frac{2GM\delta(r)}{c^2}\sqrt{\frac{B}{AB(0)}} \tag{35}$$

$$R_{\varphi\varphi} = \sin^2\theta \; R_{\theta\theta} \tag{36}$$

$$R_{tt} = -\frac{B^{''}}{2A} + \frac{B^{'}}{4A}(\frac{A^{'}}{A} + \frac{B^{'}}{B}) - \frac{B^{'}}{rA} = -\frac{2GM\delta(r)}{c^2r^2}B\sqrt{\frac{B}{AB(0)}} \tag{37}$$

It can be checked easily that the usual solutions $AB = 1$ and $B = 1 - \frac{2GM}{c^2r}$ do not satisfy (34)-(37). This point has been recognized by Narlikar too [10]. He says when we try to solve for a point mass gravitational field the condition $AB = 1$ yields to an inconsistency at $r = 0$ because this gives ${T^0}_0 = {T^1}_1$ while for a static problem we need ${T^1}_1 = 0$. He argues that it would be tempty to take the easy way out of the problem by taking that the Schwarzschild coordinates are in appropriate at $r = 0$.

Using (34) and (37), $\frac{R_{rr}}{A} + \frac{R_{tt}}{B}$ gives

$$-\frac{1}{rA}(\frac{A^{'}}{A} + \frac{B^{'}}{B}) = -\frac{4GM\delta(r)}{c^2r^2}\sqrt{\frac{B}{AB(0)}} \tag{38}$$

and

$$\frac{A^{'}}{A} + \frac{B^{'}}{B} = \frac{4GM\delta(r)}{c^2r}\sqrt{\frac{AB}{B(0)}} \tag{39}$$

Now we may combine (39) and (35) to write

$$-1 - \frac{rA^{'}}{A^2} + \frac{2GM\delta(r)}{c^2}\sqrt{\frac{B}{AB(0)}} + \frac{1}{A} = -\frac{2GM\delta(r)}{c^2}\sqrt{\frac{B}{AB(0)}} \tag{40}$$

which gives

$$-1+\frac{d}{dr}(\frac{r}{A})+\frac{4GM\delta(r)}{c^2}\sqrt{\frac{B}{AB(0)}}=0 \tag{41}$$

Integrating (41) with respect to r from $+\infty$ to r and taking that $\sqrt{\frac{B}{AB(0)}}\delta(r)=\frac{\delta(r)}{\sqrt{A(0)}}$, yields

$$-r+\frac{r}{A}+\frac{4GM(\theta(r)-1)}{c^2\sqrt{A(0)}} \quad = const. \ , \tag{42}$$

where $\theta(x)=\begin{cases} 1 & x>0 \\ \frac{1}{2} & x=0 \\ 0 & x<0 \end{cases}$.

For finding the integration constant in (42) we may impose this fact that at large distances we have $AB=1$. Then we get

$$-r+rB+0=const. \ , \tag{43}$$

and at this range B is equal to $1-\frac{2GM}{c^2r}$. By inserting this value for B in (43) the constant in (42) and (43) will be found to be equal to $-\frac{2GM}{c^2}$. Putting this in (42) results that

$$A=\frac{1}{1-\frac{2GM}{c^2r}[1+\frac{\theta(r)-1}{\sqrt{A(0)}}]} \tag{44}$$

Taking the limit of $r\to 0$, (44) gives

$$A(0)=\lim_{r\to 0}\frac{c^2r}{2GM}\sqrt{A(0)} \tag{45}$$

or

$$A(0)=\lim_{r\to 0}\left(\frac{c^2r}{2GM}\right)^2 \tag{46}$$

Considering (46), we may write (44) as

$$A=\frac{1}{1-\frac{2GM}{c^2r}-\frac{8G^2M^2}{c^4r^2}(\theta(r)-1)} \tag{47}$$

because the correction term is merely effective at $r = 0$.

Then we have

$$\frac{A'}{A} = \frac{-\frac{2GM}{c^2r^2} - \frac{16G^2M^2}{c^4r^3}(\theta(r)-1) + \frac{8G^2M^2\delta(r)}{c^2r^2}}{1 - \frac{2GM}{c^2r} - \frac{8G^2M^2}{c^4r^2}(\theta(r)-1)} \tag{48}$$

Putting (48) in (39) and using this fact that at r.h.s of it $\sqrt{\frac{AB}{B(0)}}$indeed is $\sqrt{A(0)}$ and (46) yield

$$\frac{B'}{B} = \frac{\frac{2GM}{c^2r^2} + \frac{16G^2M^2}{c^4r^3}(\theta(r)-1)}{1 - \frac{2GM}{c^2r} - \frac{8G^2M^2}{c^4r^2}(\theta(r)-1)} \tag{49}$$

Integrating (49) with respect to r from $+\infty$ to r and imposing $B(+\infty) = 1$ gives

$$B = \{1 - \frac{2GM}{c^2r} - \frac{8G^2M^2}{c^4r^2}(\theta(r)-1)\}\exp[2(\theta(r)-1)] \tag{50}$$

Using (47) and (50) we have

$$AB = \exp[2(\theta(r)-1)] \tag{51}$$

Finally our rigorous result for Schwarzschild metric is:

$$\begin{aligned} ds^2 &= c^2\{1 - \frac{2GM}{c^2r} - \frac{8G^2M^2}{C^4r^2}(\theta(r)-1)\}\exp[2(\theta(r)-1)]dt^2 \\ &\quad - \frac{1}{1 - \frac{2GM}{c^2r} - \frac{8G^2M^2}{c^4r^2}(\theta(r)-1)}dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) \end{aligned} \tag{52}$$

It can be easily verified that this metric satisfies the field equation exactly at each point even at $r = 0$. In a mathematical rigorous treatment one may truly expect that the theory of distribution should be considered [11-15]. This is not a possible task at this stage because distribution in curved space-time is not well known. Here theta and delta functions have been considered in the same manner which are commonly being used in physics literature. This work is a step toward a perfect treatment in this sense. Another point to notice is that in Newtonian gravity the potential is a continuous function even at $r = 0$ and the force is always attractive. In this result there exist a discontinuity in the metric at $r = 0$. The neighborhood of $r = 0$ that is where $r << 2M$ is a domain in which the field is very strong and therefore there is no weak field limit corresponding to it. This phenomenon is a pure general relativistic effect which has no analog in Newtonian gravity. The existence of this discontinuity means that we have an infinite repulsive force at $r = 0$ (see appendix). For more clarification we will find the tidal forces and the scalar invariant $R^{\mu}{}_{\nu\lambda\rho}R_{\mu}{}^{\nu\lambda\rho}$ for this metric in the next section. This repulsive force causes to change the ultimate fate of a free fall test particle from a collapse into intrinsic singularity to a bouncing state. The prediction of this phenomenon itself may be considered as a justification for this calculations which was rather detailed and the presented treatment for determining the form of the line element.

1 Tidal Forces

The tidal effect results in an elongation of the distribution in the direction of motion and a compression of the distribution in transverse directions. The same effect occurs in a body falling towards a spherical object in general relativity. We can gain some idea of this by considering the equation of geodesic deviation in the form [16]

$$\frac{D^2\eta^\alpha}{D\tau^2} - R^a{}_{bcd}e^\alpha{}_a v^b v^c e_\beta{}^d \eta^\beta = 0 \tag{53}$$

for the spacelike components of the orthogonal connecting vector η^a connecting two neighbouring particles in free fall.
Let the frame $e_i{}^a$ be defined in Schwarzschild coordinate as

$$\begin{array}{rcl} e_0{}^a & = & B^{-\frac{1}{2}}(1,0,0,0) \\ e_1{}^a & = & A-\frac{1}{2}(0,1,0,0) \\ e_2{}^a & = & r^{-1}(0,0,1,0) \\ e_3{}^a & = & (rsin\theta)^{-1}(0,0,0,1) \end{array} \tag{54}$$

and denote the components η^α by

$$\eta^\alpha = (\eta^1, \eta^2, \eta^3) = (\eta^r, \eta^\theta, \eta^\varphi) \tag{55}$$

Using (1) the nonvanishing components of the curvature tensor are :

$$
\begin{array}{llcll}
R^t{}_{rtr} & = -\frac{B''}{2B} & , & R^r{}_{trt} & = \frac{B''}{2A} \\
R^t{}_{\theta t\theta} & = \frac{rA'}{2A^2} & , & R^r{}_{\theta r\theta} & = R^t{}_{\theta t\theta} \\
R^t{}_{\varphi t\varphi} & = R^t{}_{\theta t\theta}\ Sin^2\theta & , & R^r{}_{\varphi r\varphi} & = R^t{}_{\varphi t\varphi} \\
R^\theta{}_{t\theta t} & = \frac{B'}{2rA} & , & R^\varphi{}_{t\varphi t} & = R^\theta{}_{t\theta t} \\
R^\theta{}_{r\theta r} & = \frac{A'}{2rA} & , & R^\varphi{}_{r\varphi r} & = R^\theta{}_{r\theta r} \\
R^\theta{}_{\varphi\theta\varphi} & = R^\varphi{}_{\theta\varphi\theta}\ Sin^2\theta & , & R^\varphi{}_{\theta\varphi\theta} & = 1-\frac{1}{A}
\end{array}
\tag{56}
$$

By using (56) and (54) , (53) gives the following

$$\frac{D^2\eta^r}{D\tau^2}+\frac{B''}{2AB}\eta^r=0 \tag{57}$$

$$\frac{D^2\eta^\theta}{D\tau^2}+\frac{B'}{2rAB}\eta^\theta=0 \tag{58}$$

$$\frac{D^2\eta^\varphi}{D\tau^2}+\frac{B'}{2rAB}\eta^\varphi=0 \tag{59}$$

Using (47), (50) and(51) we have

$$\frac{B'}{2AB}=\frac{GM}{c^2r^2}+\frac{8G^2M^2}{c^4r^3}(\theta(r)-1)-\frac{4G^2M^2}{c^4r^2}\delta(r)+\frac{\delta(r)}{A} \tag{60}$$

By (46) we have, $\frac{\delta(r)}{A}=\frac{4G^2M^2}{c^4r^2}\delta(r)$, then (60) becomes

$$\frac{B'}{2AB}=\frac{GM}{c^2r^2}+\frac{8G^2M^2}{c^4r^3}(\theta(r)-1) \tag{61}$$

Next (34) gives

$$\frac{B''}{2AB}=\frac{B'}{4AB}(\frac{A'}{A}+\frac{B'}{B})+\frac{A'}{rA^2}-\frac{2GM}{c^2r}\sqrt{\frac{B}{AB(0)}}\delta(r) \tag{62}$$

Inserting (61) , (39) and (46) in (62) yields

$$\frac{B''}{2AB} = -\frac{2GM}{c^2r^3} - \frac{16G^2M^2}{c^4r^4}(\theta(r)-1) + \frac{GM}{c^2r^2}\delta(r) \tag{63}$$

Finally putting (61) and (63) in (57)-(59) leads to

$$\frac{D^2\eta^r}{D\tau^2} = \{\frac{2GM}{c^2r^3} + \frac{16G^2M^2}{c^4r^4}(\theta(r)-1) - \frac{GM}{c^2r^2}\delta(r)\}\eta^r \tag{64}$$

$$\frac{D^2\eta^\theta}{D\tau^2} = \{-\frac{GM}{c^2r^3} - \frac{8G^2M^2}{c^4r^4}(\theta(r)-1)\}\eta^\theta \tag{65}$$

$$\frac{D^2\eta^\varphi}{D\tau^2} = \{-\frac{GM}{c^2r^3} - \frac{8G^2M^2}{c^4r^4}(\theta(r)-1)\}\eta^\varphi \tag{66}$$

(64)-(66) exhibit the usual tidal force of the Schwarzschild metric for $r \neq 0$. At $r = 0$ the correction terms are dominant and have opposite sign to the first terms.

2 Scalar Invariant

Since the metric components have no cross terms and furthermore the curvature tensors have some algebric properties, the Riemann tensor scalar invariant take a simple form as

$$R^a{}_{bcd}R_a{}^{bcd} = 2\sum_{a\neq b}(g^a aR^b{}_{aba})^2 \tag{67}$$

By using (56) we may write (67) as

$$R^a{}_{bcd}R_a{}^{bcd} = (\frac{B''}{AB})^2 + 4(\frac{A'}{rA^2})^2 + \frac{4}{r^4}(1-\frac{1}{A})^2 \tag{68}$$

Inserting (63) and (47) in (68) gives

$$\begin{aligned} R^a{}_{bcd}R_a{}^{bcd} = 4\{-\frac{2GM}{c^2r^3} - \frac{16G^2M^2}{c^4r^4}(\theta(r)-1) + \frac{GM}{c^2r^2}\delta(r)\}^2 \\ +\frac{4}{r^2}\{\frac{2GM}{c^2r^2} + \frac{16G^2M^2}{c^4r^3}(\theta(r)-1) - \frac{8G^2M^2}{c^4r^2}\delta(r)\}^2 \\ +\frac{4}{r^2}\{\frac{2GM}{c^2r^2} + \frac{16G^2M^2}{c^4r^3}(\theta(r)-1)\}^2 \end{aligned} \tag{69}$$

(69) shows that $R^a{}_{bcd}R_a{}^{bcd}$ behaves like $\frac{48G^2M^2}{c^4r^6}$ for $r \neq 0$, the same result of the Schwaezschild metric. At $r = 0$ in addition of this term there are some correction terms which all of them diverge.

3 Remark

According to singularity theorems a condition for the existence of the singularity is that $R_{ab}K^aK^b \geq 0$ for every non-spacelike vector K. Now using the four velocity (18) as a vector we obtain in this case

$$R_{ab}u^a u^b = -\frac{GM\delta(r)}{r^2}\sqrt{\frac{B}{AB(0)}} = -\frac{\delta(r)}{2r} \tag{70}$$

which is negative for $r = 0$. This means this problem does not satisfy the conditions of singularity theorems and they are not applicable in this case. This is the reason why the test particle has the chance to escape this curvature singularity.

It is noticable that from (46) we may infer that A and B^{-1} approach to positive zero as r goes to zero. This means at $r = 0$ the common convention of t as time coordinate and r as space coordinate remain valid. This is in agreement with our initial assumption that the point particle is located at $r = 0$ for ever.

We may conclude our discussion by expressing that there is no trace of those conceptual problems which was raised by Narlikar in this form of the metric.

4 Appendix

The geodesic equations of this metric are the following, where p is a parameter describing the trajectory.

$$\frac{d^2r}{dp^2} + \frac{A'}{2A}\left(\frac{dr}{dp}\right)^2 - \frac{r}{A}\left(\frac{d\theta}{dp}\right)^2 - \frac{r\sin^2\theta}{A}\left(\frac{d\varphi}{dp}\right)^2 + \frac{B'}{2A}\left(\frac{dt}{dp}\right)^2 = 0 \tag{71}$$

$$\frac{d^2\theta}{dp^2} + \frac{2}{r}\frac{d\theta}{dp}\frac{dr}{dp} - \frac{\sin^2\theta}{2}\left(\frac{d\varphi}{dp}\right)^2 = 0 \tag{72}$$

$$\frac{d^2\varphi}{dp^2} + \frac{2}{r}\frac{d\varphi}{dp}\frac{dr}{dp} + 2\cot\theta\,\frac{d\theta}{dp} = 0 \tag{73}$$

$$\frac{d^2t}{dp^2} + \frac{B'}{B}\frac{dr}{dp}\frac{dt}{dp} = 0 \tag{74}$$

Since the field is isotropic, we may consider the orbit of the test particle to be confined to the equatorial plane,that is $\theta = \frac{\pi}{2}$. The equation (72) immediately is satisfied and we can forget about θ as a dynamical variable. Then equation (73) gives:

$$\frac{d^2\varphi}{dp^2} + \frac{2}{r}\frac{d\varphi}{dp}\frac{dr}{dp} = 0 \quad or \quad \frac{d}{dp}\left[\ln\left(\frac{d\varphi}{dp}\right) + \ln r^2\right] = 0 \tag{75}$$

Integrating (75) with respect to p leads to:

$$\frac{d\varphi}{dp} = \frac{J}{r^2} \tag{76}$$

where J is a constant. Dividing (74) by $\frac{dt}{dp}$ gives

$$\frac{d}{dp}\left[\ln\left(\frac{dt}{dp}\right) + \ln B\right] = 0 \quad or \quad \frac{dt}{dp} = \frac{1}{B} \tag{77}$$

The related constant of integration has been absorbed into the definition of p. Inserting (76) into (71) we obtain

$$\frac{d^2r}{dp^2} + \frac{A'}{2A}\left(\frac{dr}{dp}\right)^2 - \frac{J^2}{r^3 A} + \frac{B'}{2AB^2} = 0 \tag{78}$$

or

$$\frac{d}{dp}\left[A(r)\left(\frac{dr}{dp}\right)^2 + \frac{J^2}{r^2} - \frac{1}{B}\right] = 0 \tag{79}$$

By integrating (79) with respect to p we get

$$A(r)\left(\frac{dr}{dp}\right)^2+\frac{J^2}{r^2}-\frac{1}{B}=-E \tag{80}$$

where E is a constant of integration and if the test particle is going to be moveless at infinity we must have $E=1$. Equation (80) for a geodesic with $d\varphi=0$ i.e. $J=0$ becomes

$$A(r)\left(\frac{dr}{dp}\right)^2-\frac{1}{B}=-1 \tag{81}$$

Inserting (81) in (78) with $J=0$ gives

$$\frac{d^2r}{dp^2}+\frac{1}{2AB}\left(\frac{A'}{A}+\frac{B'}{B}\right)-\frac{A'}{2A^2}=0 \tag{82}$$

and inserting (39) in (82) gives

$$\frac{d^2r}{dp^2}+\frac{1}{2}\frac{d}{dr}\left(\frac{1}{A}\right)+\frac{2GM\delta(r)}{c^2r\sqrt{ABB(0)}}=0 \tag{83}$$

and at last using (47) and (50) in (83) yields the final conclusion

$$\frac{d^2r}{dp^2}=-\frac{GM}{c^2r^2}-\frac{8G^2M^2}{c^4r^3}(\theta(r)-1)+(\frac{4G^2M^2}{c^4r^2}-2e)\delta(r) \tag{84}$$

In the r.h.s of (84) the first term exhibits the common attractive force and the other terms represent point interactions which act at $r=0$. Indeed at the origin the third is the dominant repulsive force.

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